and (28.2) and (33.2) may be written respectively as

$$\eta_c = 1 - \frac{0.77}{\eta_0}\frac{1}{\Psi_0}\left(\frac{\lambda_0}{\lambda}\right)^{5/2} \quad .. \quad .. \quad (38)$$

and, from (36),

$$P_L = \frac{V_{OB}^2\eta_0}{R_{BO}}\left(\frac{\lambda}{\lambda_0}\right)^2\left[1 - \frac{0.77}{\eta_0}\frac{1}{\Psi_0}\left(\frac{\lambda_0}{\lambda}\right)^{5/2}\right] \quad .. \quad .. \quad .. \quad (39)$$

The foregoing analysis is the general case for the transfer of energy from an electron stream to a field and then to a load, and includes the more familiar "lumped circuit" concepts as special approximate cases.

(To be continued)

(Bibliography will be included at end of Part III of the article.)

H.F. RESISTANCE AND SELF-CAPACITANCE OF SINGLE-LAYER SOLENOIDS

By R. G. Medhurst, B.Sc.

(Communication from the Staff of the Research Laboratories of The General Electric Company, Limited, Wembley, England.)

(Concluded from page 43 of the February issue.)

9. Self-capacitance of Single-layer Coils.

9.1. The self-capacitance of each coil, including capacitance due to leads, has to be added to the parallel capacitance reading of the twin-T. It is a small correction, usually less than 1 per cent. The original intention was to use Palermo's formula for self-capacitance[14], this being available in abac form[18] and hence readily made use of. However, for the closely-spaced coils, a noticeable variation with frequency started to appear in the calculated values of inductance (which should be consistent to better than ½ per cent), so it was decided that an attempt should be made to find out whether Palermo's formula did in fact agree with experiment, and, if there was a substantial disagreement, whether an empirical formula could be substituted.

What was required was a set of formulae, or, preferably, a set of curves from which the self-capacitance of a particular coil could be quickly and easily read off, say to 20 per cent or better. Since the capacitance of the leads is of the same order of magnitude as that of the coil, it was first necessary to find out whether the lead capacitance could be specified by some quantity which would be additive algebraically to the self-capacitance of the coil.

The simplest hypothesis is that the "live" lead can be treated as an isolated straight vertical wire; that is to say, that (1) the fact of its being bent, and (2) the proximity of the coil to the upper end have a negligible effect on its capacitance. This we shall show to be correct, to the degree of approximation we require.

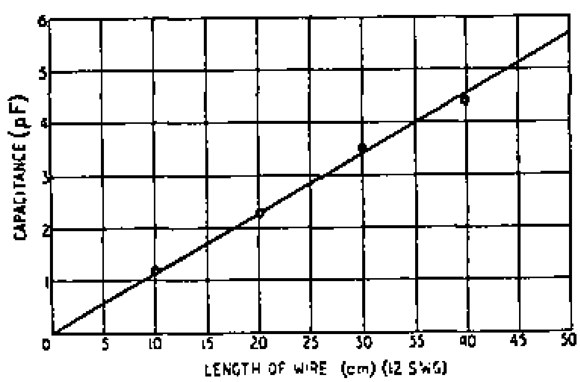

Fig. 4. *Variation with length of capacitance of vertical 12 gauge copper wire.*

9.2. The capacitances of a number of copper wires of various lengths and diameters, from 10 to 40 cm in length and from 12 S.W.G. to 44 S.W.G., were measured at 200 kc/s, the wires standing vertically upright with their lower ends in the live terminal of a Cambridge Capacity Meter. Over this range of length, the capacitances of each wire gauge were quite closely proportional to their lengths (see, for example, Fig. 4).

The capacitances, measured in this way, of 25-cm lengths of wire of various gauges are plotted against the wire diameter in Fig. 5. In Fig. 6 capacitance is plotted against length for a number of wire gauges.*

We may readily show that bending of the wire and alteration of its position relative to earth make no large difference to the measured capacitance. A 25-cm length of No. 12 S.W.G. copper wire was measured in a vertical position, as before. Its capacitance was 2.9 pF. Now, a piece of brass sheet, 35 cm × 15 cm, was attached to the earth terminal of the capacitance meter, so that it formed a horizontal earth adjacent to the wire in the live terminal. The wire was bent over so that about 2/3 of its length was horizontal and about 6 cm above the brass sheet. The capacitance was now 3.0 pF. Even when the wire was brought to within about 2 cm of the earth plate, the capacitance reading only rose to 3.6 pF. Finally, the wire was screwed into the meter terminal at its centre, the two ends being bent up to about 45° to the horizontal. The capacitance reading was now 3.1 pF.

Some of these measurements were repeated on the twin-T, at frequencies up to 20 Mc/s, and close agreement was obtained. In addition, some observations were made on the effect of the proximity of an adjacent vertical earth lead, screwed into the earth terminal of the twin-T. It was found that there was no measurable increase in capacitance until the earth lead was brought to within one or two centimetres of the live lead.

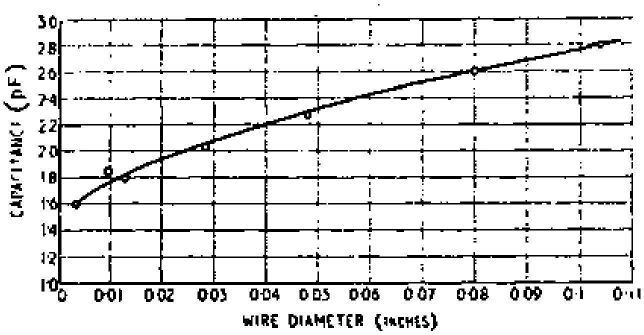

Fig. 5. *Capacitance of 25-cm lengths of vertical copper wire of various diameters.*

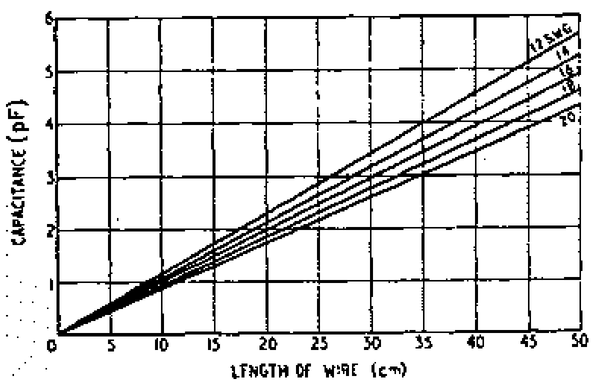

Fig. 6. *Variation of capacitance with wire length for vertical copper wires of various gauges.*

9.3. Before we discuss the effect of the proximity of the coil on the capacitance of the "live" lead, we have to describe the method used for measuring the capacitance of the whole coil-lead assembly.

The standard technique for making self-capacitance measurements on coils was originally suggested by G. W. O. Howe[17]. The square of the wavelength is plotted against the added parallel capacitance necessary to resonate the coil. The points so obtained should lie on a straight line, which is produced to meet the capacitance axis, making a negative intercept which is numerically equal to the self-capacitance. The present method is a modification of this, making use of the large range (1,000 pF) of the main tuning capacitor of the twin-T and its fine graduation (0.2 pF per division). A measurement is carried out at the frequency at which the coil resonates with about 1,000 pF. About half a dozen additional measurements are now required, the first at about four times this frequency and the remainder at frequencies increasing in steps of 2 or 3 Mc/s.

Now, if we know the self-capacitance (including lead capacitance), we can calculate the inductance from any one of these measurements, since the coil is resonating with its self-capacitance plus the added capacitance. To obtain a given accuracy of inductance

* It is interesting to note that, over these ranges of length and diameter, the theoretical expression, given originally by G. W. O. Howe (see ref. 19: also ref. 12 p. 116), for capacitance of a straight vertical wire above a plane earth is very roughly linear with respect to the length of wire. Our experimental points, however, fit more closely to a straight line than to this theoretical curve. The theoretical curve for 12-gauge wire intersects the experimental straight line at the 45-cm length point and is about 0.4 pF above at a length of 10-cm. In the 20-gauge case, the theoretical curve falls above the experimental line throughout the range, the maximum deviation being about 0.3 pF.

we need to know the self-capacitance less accurately as the added capacitance becomes higher. In particular, if we make use of the measurement involving an added capacitance of about 1,000 pF, quite a rough value of the

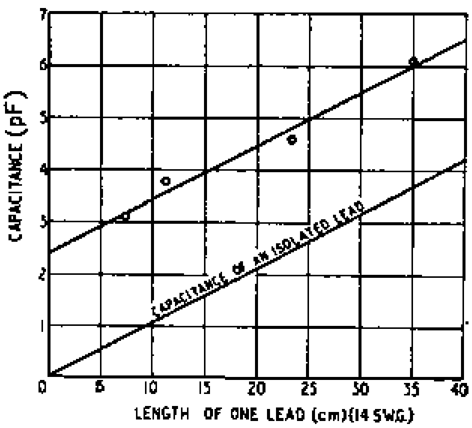

Fig. 7. *Apparent self-capacitance of a coil with various lengths of leads.*

self-capacitance (which is not usually greater than 5 pF) will yield an inductance value of very high accuracy. The rough value is derived from the 1,000-pF measurement and the measurement involving the lowest added capacitance. In practice, we do not actually work out this self-capacitance correction, the inductance being obtained directly from the formula

$$L = \frac{0.02533}{C_2 - C_1}\left[\frac{1}{f_2^2} - \frac{1}{f_1^2}\right]$$

C_1 and C_2 (pF) being the added capacitances at frequencies f_1 and f_2 Mc/s) respectively.

Finally, using this value of inductance we can calculate the self-capacitance, at each of the frequencies of measurement after the first, from the formula

$$C_0 = \frac{0.02533}{Lf^2} - C$$

where C is the added capacitance at frequency f.

As an example of this method, coil No. 32 had 38 turns of 20 S.W.G. copper wire, mean diameter being 5.10 cm, overall length 4.79 cm, spacing ratio 0.720. Self-capacitance measurements took the form shown in Table IV.

The live lead consisted of 10 cm of 14 S.W.G. copper wire. Thus, a lead capacitance of 1.03 pF (independent of frequency) has to be subtracted from each of the readings in Table IV, to give the actual self-capacitance of the coil (see below, Sections 9.4 and 9.6). The mean self-capacitance now becomes 2.30 pF.

It appears, from these results, that the reactance of this coil can be represented closely, over quite a wide frequency range up to and beyond the self-resonant frequency, by a fixed inductance in parallel with a fixed capacitance. This is true for all the coils measured, no evidence being found for the suggestion sometimes made (e.g., ref. 12, p. 84, footnote) that self-capacitance is lower at the self-resonant frequency of the coil than at frequencies much less than this.

9.4 Now we can return to the question of the effect of the lead capacitance on the total measured capacitance.

TABLE IV.

Frequency Mc/s	C_1 (pF)	C_2 (pF)	C	L		C_0
			$C_2 - C_1$ (pF)	Inductance (μH)	$\frac{0.02533}{Lf^2}$ (pF)	$\frac{0.02533}{Lf^2} - C$ (pF)
0.72	100	1076.0	976.0	49.89		
3.0	100	153.2	53.2		56.4	3.2
6.0	200	210.7	10.7		14.1	3.4
8.0	150	154.6	4.6		7.9	3.3
12.0	200	200.15	0.15		3.52	3.4
15.0	300	298.95	−1.05		2.25	3.3
18.0	200	198.2	−1.8		1.57	3.4
					Mean.	3.33

A coil was constructed (39 turns of 20 gauge wire, mean diameter 5.08 cm, overall length 4.70 cm) having 14 gauge leads each 35 cm in length, inclusive of the portion bent over near the twin-T terminals. The parallel capacitance of coil plus leads was measured as just described, and the measurements repeated when the leads were shortened to 23.5, 11.5 and 7.5 cm.

The results are plotted, in Fig. 7, against the length of the live lead. If the lead capacitance adds algebraically, without modification, on to the coil self-capacitance, these points should lie on a straight line

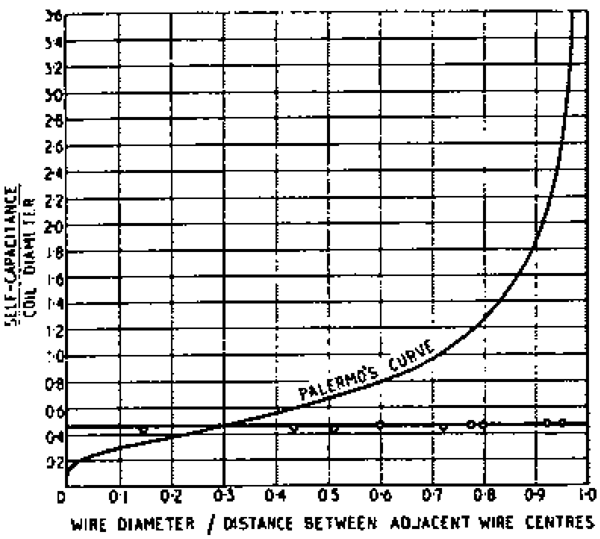

Fig. 8. *Comparison between Palermo's formula and measured self-capacitances of coils having length/diameter = 1 approximately.*

parallel to the 14 S.W.G. line of Fig. 6. By the method of least squares, the best fitting straight line has been drawn among these points, which deviate from it by not more than 5%. This line, it will be seen, is very closely parallel to the 14 S.W.G. line.

9.5. We are now in a position to deal with Palermo's self-capacitance formula. Previous work [14–17,20] has established that the self-capacitance of a single-layer coil (C_0) is directly proportional to the coil diameter. It is also independent of the number of turns, provided this number is not too small. The remaining quantities upon which C_0 might depend are the ratio of coil length to diameter, the wire diameter (d) and the spacing of the turns (s). Investigators before Palermo had assumed that C_0 was independent of d and s. Palermo asserted that C_0 varied with d and s according to the following relation:

$$C_0 = \frac{\pi D}{3.6 \cosh^{-1} s/d}$$

where D is the coil diameter (cm).

This result, independent of the length of the coil, was supposed to hold for coils whose length/diameter ratio was equal to or less than 1.

Fig. 8 shows measured values of the ratio C_0/D for nine coils having diameters ranging from 2.6 to 6.4 cm and spacing ratios (d/s) from 0.15 to 0.95. Wire gauges used range from 18 to 30 S.W.G. All the coils were wound with bare wire on grooved Distrene formers except two, with values of d/s equal to 0.947 and 0.919, which were wound respectively with single-silk-covered and double-silk-covered wire on ungrooved Distrene rod, the turns being as close together as possible. Values of length/diameter were all about 1, ranging from 0.94 to 1.49. Each coil was measured as described above (see Table IV), lead capacitances being subtracted. In Fig. 8, Palermo's theoretical expression for C_0/D is plotted against d/s, and the experimental values are plotted on the same scale. To better than 5%, the measured values fit the expression

$$C_0 = 0.46\ D,$$

being independent of the spacing ratio. These observed values show a tendency to increase slightly with increasing proximity of turns, but this increase was of the order of magnitude of the experimental error anticipated, and it was not thought than any useful conclusions could be drawn.

It has to be pointed out that this experi-

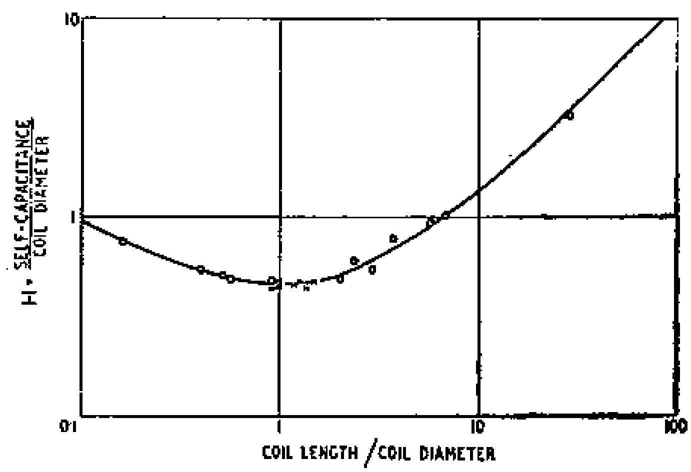

Fig. 9. *Variation of self-capacitance with coil length (one end of coil earthed).*

mental demonstration of the lack of dependence of self-capacitance on the spacing of turns contradicts not only Palermo's theory but also some experimental confirmation which he brought forward (see Section 9.7 below.) Consequently, it seems advisable to remark that no investigators other than Palermo have found a measurable variation with turn spacing. J. C. Hubbard [16], for example, says:

"There is no evidence that the variation of ratio of pitch to diameter of wire has a measurable effect on the distributed capacity in the region studied, though some effect is to be expected for coils of a smaller number of turns than those studied here." Hubbard's minimum number of turns was 35, and he worked down to a length/diameter of about 0.2; i.e., his coils are "short" enough for Palermo's formula to be applicable.

9.6. C_0 having been shown to be substantially independent of d/s, the final step is to find the variation of C_0 with the length/diameter ratio. Fig. 9 shows the results of a series of measurements on coils whose length/diameter ranged from 29.2 to 0.163. Diameters ranged from 0.675 to 6.36 cm, and numbers of turns from 10 to about 636. All the coils were those which had been used for h.f. resistance measurements, except the two with the greatest and smallest ratios of length/diameter. The former was wound with about 636 turns of 34 gauge wire, double-silk-covered, on a ¼-in Distrene former, and the latter with 10 turns of 20 gauge wire, double-silk-covered, on a 2½-in Distrene former.

It appears that, in the commonly occurring case when one end of the coil is at earth potential, we can write down the self-capacitance in the form

$C_0 = HD$ picofarads, where D is in centimetres.

H depends on the length/diameter ratio only. The table of values of H which follows is based on the curve of Fig. 9. The use of these values, with the appropriate lead correction, should give results accurate to 5% or better.

TABLE V.

Length/Diameter	H	Length/Diameter	H	Length/Diameter	H
50	5.8	5.0	0.81	0.70	0.47
40	4.6	4.5	0.77	0.60	0.48
30	3.4	4.0	0.72	0.50	0.50
25	2.9	3.5	0.67	0.45	0.52
20	2.36	3.0	0.61	0.40	0.54
15	1.86	2.5	0.56	0.35	0.57
10	1.32	2.0	0.50	0.30	0.60
9.0	1.22	1.5	0.47	0.25	0.64
8.0	1.12	1.0	0.46	0.20	0.70
7.0	1.01	0.90	0.46	0.15	0.79
6.0	0.92	0.80	0.46	0.10	0.96

J. C. Hubbard[16] remarked: "... we apparently have two quite independent factors" (determining the self-capacitance of coils), "one predominating greatly in very short coils, the other, in very long coils." It is an interesting confirmation of this suggestion that the experimental results of Fig. 9 and Table V can be fitted quite closely (to 2 or 3%) by an expression of the form

$$H = 0.1126\,\frac{l}{D} + 0.08 + \frac{0.27}{\sqrt{l/D}}$$

The first numerical factor follows from Nagaoka's inductance formula for long coils and the experimental fact that the self-resonant wavelength for long coils equals twice the length of winding (see below). The other two factors are empirical.

A few additional measurements were made on some two-turn and single-turn coils. A coil of two turns of closely-spaced 18 S.W.G. double-silk-covered wire, diameter 6.47 cm, length/diameter 0.042 gave an H value of 1.53, which is quite close to the value, 1.40, calculated from the expression above. Another two-turn coil, of closely-spaced double-silk-covered 40 gauge wire, diameter 4.46 cm, length/diameter 0.0067, gave the low H value of 0.96. The lead correction is uncertain in both these cases, the assumptions about the live and the earth leads needing modification when the length of the lead becomes comparable with the winding length. It seems from these results that the curve of Fig. 9 can be extrapolated to a length/diameter of about 0.05, even when the number of turns is only two, but that there is a considerable falling off thereafter. A one-turn coil (14 S.W.G., mean diameter 23.9 cm, length/diameter 0.0084) departed even more from the trend of the curve in Fig. 9, the H value being only 0.23.

As an example of the use of Fig. 6 and 9, we may take the coil dealt with in Table III. Ratio of length to diameter was 1.375, and mean diameter was 5.10 cm. Hence, from Fig. 9,

self-capacitance of coil $= 5.10 \times 0.47 = 2.4$ pF.

The leads were of 14 S.W.G., the length of each was 9.5 cm. Hence, from Fig. 6,

capacitance of live lead $= 1.0$ pF

Thus, total capacitance $= 2.4 + 1.0$ pF $= 3.4$ pF.

9.7 The wide discrepancy between

Palermo's results and the present work make it desirable to say something about the theoretical basis of the expression put forward by Palermo.

What is called the "self-capacitance" of a coil will actually be a composite quantity, and the components will not necessarily be mutually dependent. It is convenient, to begin with, to divide coil self-capacitance into two parts, the "internal" and the "external" capacitances. When a current flows through the coil, each turn is at a different mean potential from every other turn. Consequently, there will be capacitances between each pair of turns (modified by the presence of the other turns between or on either side of the particular pair.). We shall call the effective parallel capacitance, across the whole coil inductance, the "internal" capacitance; it is formed by summing all these capacitances between turns, each taken across the appropriate part of the inductance.

Furthermore, each turn will be at a mean potential different from that of the earth, so that each turn will show a capacitance to earth. The effective parallel capacitance formed by summing these capacitances to earth we shall call the "external" capacitance.

It will be apparent that if the external and internal capacitances are comparable in magnitude, the apparent self-capacitance will be different when neither end of the coil is earthed, since the external capacitance will then not appear directly across the terminals of the coil. Hence, the present results, which are all for coils earthed at one end, may not be applicable to coils both ends of which are above earth potential.

Palermo further divides the internal capacitance into two portions, the capacitance between adjacent turns and the capacitance between turns which are not adjacent. He assumes that almost the whole of the self-capacitance is made up of the portion of the internal capacitance between adjacent turns: that is to say, he asserts that the capacitance between non-adjacent turns will be negligible, and he fails to mention the external capacitance.

Now, in spite of having neglected what may be a large part of the total self-capacitance, he predicts values which, for closely spaced coils, are very much larger than the values we have measured. The reason for this over-estimate is not too difficult to see.

Palermo derives his capacitance between adjacent turns from the formula for the capacitance between long parallel cylinders, diameter d and separation of centres s, which he quotes in the form

$$C = \frac{1}{3.6 \cosh^{-1} s/d} \text{ picofarads/cm.}$$

When s/d approaches 1, that is to say, when the cylinders are very close, this expression approaches infinity. However, when the turns of a coil are very close the self-capacitance does not approach infinity, and the reason for the discrepancy appears to be that what we have to concern ourselves with is the effective current-carrying path and not the whole of the cross section of each turn.

When high-frequency current flows through an isolated wire, the current tends to be concentrated near the surface. When the wire is bent into the form of a coil, the current tends, further, to flow round the inner surface of the coil. Finally, the effect of the adjacent turns is to cause the current to withdraw from the portions of the wire nearest to these turns. Thus, even when the turns are very close the effective current-carrying paths are still comparatively remote from each other.

Thus, the capacitance between adjacent turns will be less than that predicted by Palermo. The fact that self-capacitance is substantially independent of spacing of turns suggests that the part of the self-capacitance considered by Palermo is actually negligible.

The question of the validity, or otherwise, of Palermo's formula is complicated by the existence of some measurements (on coils earthed at one end) which he brings forward in support of his theory. It is difficult to say much about these measurements, except that they are closely in agreement with Palermo's formula, and consequently, when the turns are closely spaced, they are very different from other published results on similar coils. The discrepancy is drastically illustrated by Palermo's coil No. 9, which had a diameter of 10.40 cm and a length of 9.65. The number of turns was 28, the wire diameter 0.326 cm and the spacing ratio 0.94. The coil was measured at a "high frequency"; i.e., at something below, but of the order of magnitude of the self-resonant frequency. From the curve of Fig. 9 we would predict a self-capacitance of

4.8 pF. Palermo's formula gives 20.5 pF. The measured value he gives as 20.0 pF.

Palermo's measured coils fall into two groups. Seven of them, with spacing ratios between 0.3 and 0.8 were measured by the Bureau of Standards. Over this region of spacing ratio, Palermo's "proximity effect" is not too pronounced. The measured values were all between 1 and 3 picofarads larger than the values that would be predicted from our present work. Palermo makes no mention of a correction for leads and terminals, and possibly this accounts for the discrepancy. The remaining twelve coils were measured by Palermo himself, and it is among these that we find the capacitances (such as the one already quoted) which are so greatly different in magnitude from our results.

9.8. We have seen that, so far as self-capacitance is concerned, a single-layer coil behaves very closely like a cylindrical current sheet. *It is well* known that this is also true of the inductive part of its reactance. If we combine these two current-sheet formulae we might expect to deduce some simple expression, depending on the coil geometry, for the self-resonant frequency.

Our measurements have given a self-capacitance expression in the form

$C_0 = HD$, where H is a quantity dependent on the length/diameter only.

The Nagaoka expression for the inductance, L_0, may be written in the form

$L_0 = Kn^2D$, where K is dependent on the length/diameter only.

Now, if we call λ (cm) the self-resonant wavelength, we have

TABLE VI.

Length/Diameter	N	Length/Diameter	N	Length/Diameter	N
50	2.0	5.0	2.3	0.70	3.8
40	2.0	4.5	2.4	0.60	4.0
30	2.0	4.0	2.4	0.50	4.3
25	2.0	3.5	2.5	0.45	4.5
20	2.0	3.0	2.5	0.40	4.8
15	2.1	2.5	2.6	0.35	5.0
10	2.1	2.0	2.7	0.30	5.4
9.0	2.1	1.5	2.9	0.25	5.8
8.0	2.2	1.0	3.4	0.20	6.3
7.0	2.2	0.90	3.5	0.15	7.1
6.0	2.3	0.80	3.6	0.10	8.3

$\lambda = 2\pi c\sqrt{L_0C_0}$ where c (cm/sec) is the velocity of electro-magnetic radiation,

$= 2\pi c\sqrt{HKn^2D^2}$

$= 2\pi c\, nD\sqrt{HK}$

$= Nl$ where N is dependent on the length/diameter only and l is the total length of wire.

Values of N, worked out from the inductances and self-capacitances of the coils previously measured, are plotted against length/diameter in Fig. 10. Table VI gives values of N and has been worked out from Table V and Nagaoka's values of K.

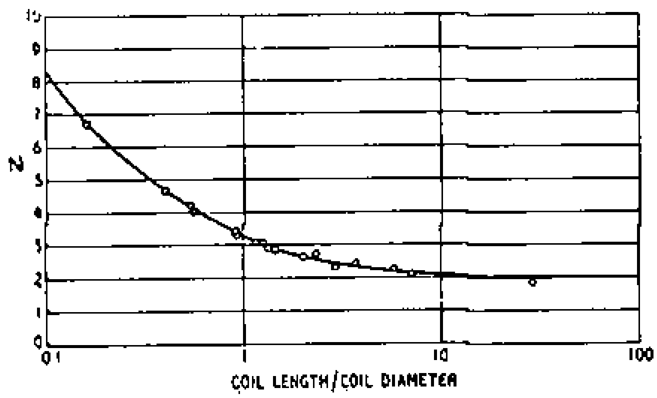

Fig. 10. *Wavelength at self-resonant frequency equals N × total length of wire.*

10. Frequency Correction.

The measured values of ϕ (ratio of the h.f. resistance of the coil to the resistance of the straightened wire at the same frequency) are mostly for frequencies such that z (see list of symbols) has values between 8 and 20. Though these frequencies are to be regarded as "high" according to our previous definition of "high frequency," (i.e., frequency for which $z>7$), ϕ will still, to some small extent, be frequency dependent. So that the measured values shall be comparable among themselves, it will be advantageous to apply a frequency correction such that the corrected ϕs correspond to the same value of z. If we choose infinity as this standard z value the corrected ϕs can be compared directly with Butterworth's "high-frequency" table, which is supposed to apply at infinitely large z. Since, as we shall see, the frequency correction is small, we may still use the corrected ϕ values at the orders of frequency commonly encountered.

It is unfortunate that exact measurements on coil resistance are almost as scarce at low as at high frequencies. Consequently, it has not been found possible to deduce from previous work an experimental frequency correction to convert the present measurements from "high" to "infinite" frequency. Tentatively, a correction formula was used based on Butterworth's theoretical considerations, modified in the light of the present results. The formula in question is

$$\Delta\phi = \frac{1}{8G}(\phi_{exp} - 2\alpha)$$

ϕ_{exp} being the measured value of ϕ, and G and α being quantities due to Butterworth (see, e.g., ref. 12, pp. 78 and 79).

The correction did not usually exceed 2%. It may be either positive or negative. In deriving $\Delta\phi$, the general form of Butterworth's resistance formula is assumed; i.e.,

$$\frac{\text{a.c. resistance}}{\text{d.c. resistance}} = \alpha H + kG$$

where the first term represents the losses due to the currents in the wires, and the second the losses due to the field of the whole coil. H and G are functions of z only, being given for large z by $\frac{\sqrt{2}\,z + 1}{4}$ and $\frac{\sqrt{2}z - 1}{8}$ respectively (the value of z chosen for each coil being that corresponding to the mean working frequency). α depends on the spacing ratio of the turns, and k on the spacing ratio and the dimensions of the coil.

Now, we have seen previously (Section 3.2) that the Butterworth theory is most open to suspicion in that part of it which deals with losses due to the "mean transverse field." The effect of these losses, in the theory, is to cause k, at infinitely high frequency, to have very high values, especially for close spacing. This is the effect that is not confirmed by the present measurements. So, to derive a frequency-correction formula, we shall assume that k has some value which does not vary with frequency (z being sufficiently high) and, eliminate k between the expressions for ϕ at the frequency of measurement and at infinite frequency. α we may take, according to the theory, as being also very nearly invariable with frequency.

When z approaches infinity, we have

$$\phi = \alpha + \frac{k}{2}$$

$$\text{Also, } \phi_{exp} = \frac{\text{a.c. resistance}}{\text{d.c. resistance}} \cdot \frac{1}{\sqrt{2}z/4} = 4\,\frac{\alpha H + kG}{(8G + 1)}$$

and hence, eliminating k from the expressions for ϕ and ϕ_{exp} and using the relation $2H = 1 + 4G$, we obtain the required expression for ϕ; i.e.,

$$\phi = \phi_{exp} + \frac{1}{8G}(\phi_{exp} - 2\alpha)$$

We shall see later that the argument for assuming k to be substantially independent of frequency, when z is high enough, is not complete, because we have only given reasons for rejecting that part of Butterworth's theory which applies to *high*-frequency coil resistance. We shall consider the low-frequency case in Section 14.

11. Effect of the Proximity of the Twin-T Top.

It was thought that an additional correction might be necessary for losses due to the proximity of the metal top of the twin-T. To ascertain the order of magnitude of this effect, a coil (48 turns of 20 gauge d.s.c. wire, mean diameter 2.70 cm, length/diameter 1.82, d/s 0.89) was measured a number of times, the leads being progressively shortened until the distance of the coil from the twin-T terminals was about its own diameter. The coil was then about 2 diameters above the twin-T top.

There was no significant variation in the

TABLE VII.

Length of each lead cm	Total Resistance ohms	Temperature °C	Total Resistance at 20° C ohms	Resistance of leads (20° C) ohms	Resistance of coil (20° C) ohms
40	$1219 \times 10^{-6}\sqrt{f}$	24	$1210 \times 10^{-6}\sqrt{f}$	$25 \times 10^{-6}\sqrt{f}$	$1185 \times 10^{-6}\sqrt{f}$
15.5	1204 ,,	24	1195 ,,	10 ,,	1185 ,,
8	1212 ,,	25	1200 ,,	5 ,,	1195 ,,
8	1218 ,,	26.5	1203 ,,	5 ,,	1198 ,,
4.5	1211 ,,	27	1195 ,,	3 ,,	1192 ,,

measured resistances, their spread being about 1 per cent. The results are given in Table VII.

The two 8-cm measurements were carried out on successive days. The length of each lead includes the right-angle bend at the twin-T terminal, so that in the case of the last measurement the coil was about 2.5 to 3 cm above the terminal.

12. Results of Measurements.

After all these corrections have been applied, we are left with a set of experimental values of ϕ for various non-integral values of coil length/diameter and d/s. These have to be reduced to a table with the same intervals as those of Table I.

We are assisted in this process by remembering that the error in Butterworth's values has been assumed to be due to excessive weight being given to the transverse field losses. If we work out Butterworth's formula again neglecting the transverse field we obtain another table whose entries are all less than those in the Butterworth table, except for the column corresponding to infinite length/diameter. In this case, the transverse field has disappeared.

Our experimental values all lie between these two sets of values. Consequently, we shall take the case where these two sets of values are equal, i.e. the extreme right-hand column, as the limiting case of our empirical table. This is convenient, because it is not possible to measure coils whose length/diameter is infinite.

Further, measurements on a few coils whose length/diameter ratio was about 8, showed that the experimental values of ϕ were within 1 or 2 per cent. of the Butterworth values when the transverse field is neglected. Since transverse-field effects are less appreciable as the length/diameter ratio increases, we may use this result to fill in the 8 and 10 length/diameter columns.

Measurements on several coils having values of $d/s = 0.2$ and 0.3, with length/diameter ranging from 0.5 to 4, showed that Butterworth's values for these two rows are confirmed by the experimental results. This was used to fill in the three lowest rows, it being assumed that the bottom row, the values in which are close to those for a straight wire, could safely be taken as following Butterworth.

It may be pointed out that this agreement with Butterworth's figures, over the region in which Butterworth's theory might be expected to hold, constitutes indirect evidence of the reliability of the measurements.

There remains the most important portion of the table, that is, the top left-hand quadrant. In general, due to difficulties in accurate grooving, the spacing ratios were not exact multiples of 0.1. However, the spacing ratios of the coils which had been constructed to have $d/s = 0.6$, turned out to be very close to the value aimed at. A smooth curve could thus be drawn through their ϕ values, giving the sixth row from the bottom. By extrapolating the values of coils having d/s about 0.5 and 0.7, using this $d/s = 0.6$ row and then drawing smooth curves, the adjacent rows were obtained, and similarly for the rest of the table.

The final result is Table VIII. The values for $d/s = 1$ are obtained by extrapolation. So are the values for the two left-hand columns.

In general, the experimental points deviate

TABLE VIII.

d/s	Coil Length/Coil Diameter											
	0	0.2	0.4	0.6	0.8	1.0	2	4	6	8	10	∞
1.0	5.31	5.45	5.65	5.80	5.80	5.55	4.10	3.54	3.31	3.20	3.23	3.41
0.9	3.73	3.84	3.99	4.11	4.17	4.10	3.36	3.05	2.92	2.90	2.93	3.11
0.8	2.74	2.83	2.97	3.10	3.20	3.17	2.74	2.60	2.60	2.62	2.65	2.81
0.7	2.12	2.20	2.28	2.38	2.44	2.47	2.32	2.27	2.29	2.34	2.37	2.51
0.6	1.74	1.77	1.83	1.89	1.92	1.94	1.98	2.01	2.03	2.08	2.10	2.22
0.5	1.44	1.48	1.54	1.60	1.64	1.67	1.74	1.78	1.80	1.81	1.83	1.93
0.4	1.30	1.29	1.33	1.38	1.42	1.45	1.50	1.54	1.56	1.57	1.58	1.65
0.3	1.16	1.19	1.21	1.22	1.23	1.24	1.28	1.32	1.34	1.34	1.35	1.40
0.2	1.07	1.08	1.08	1.10	1.10	1.10	1.13	1.15	1.16	1.16	1.17	1.19
0.1	1.02	1.02	1.03	1.03	1.03	1.03	1.04	1.04	1.04	1.04	1.04	1.05

Experimental values of the ratio of the high-frequency coil resistance to the resistance at the same frequency of the same length of straight wire.

from the smoothed curves by 1 or 2 per cent. In three cases the deviation is as high as 3 per cent.

It has already been pointed out (Section 5) that, from physical considerations, it becomes increasingly difficult to construct coils fulfilling the various criteria of Butterworth's h.f. resistance table as one approaches the extreme left-hand side of the table. The difficulty becomes acute in the bottom left-hand quadrant. To cover this region, it was necessary to sacrifice the condition that z should be high. Thus, coil No. 41, which had a mean diameter of 6.24 cm, length/diameter of 0.542, and d/s of 0.256, had to be wound with 30 gauge wire, and z was only 2.79. The correction for frequency was now about 17 per cent. This, however, is not too alarming because in this region Butterworth's formulae predict values of α and k (in the Butterworth expression for ϕ, given above) which are almost independent of frequency for large z.

The entries of Table VIII are shown graphically in Fig. 11. For closely spaced wires, there is a critical value when the length/diameter ratio is about 1. This may have some connection with the parallel phenomenon observable in the case of the self-capacitance (see Fig. 9).

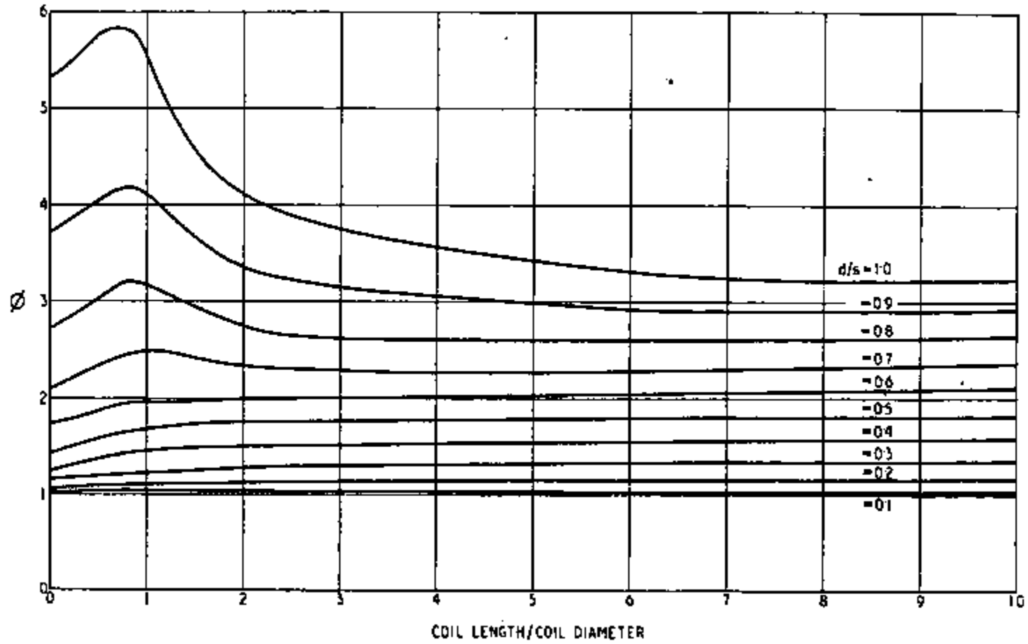

Fig. 11. *Variation of θ with spacing ratio and length/diameter ratio.*

$$\theta = \frac{\text{h.f. resistance of coil}}{\text{h.f. resistance of same length of straight wire at same frequency}}$$

13. Variation of Q with Coil Shape.

It is not very easy to judge coil performance from figures connected with the h.f. resistance. Normally, we are concerned with coil efficiency, which may best be defined by its Q value at a particular frequency.

It is well-known that Nagaoka's inductance formula may be used, with an error of not more than 5%, up to quite high frequencies. In fact, in the case of the coils used in the present series of measurements, if we assume a constant self-capacitance the inductive part of the reactance agrees closely with Nagaoka's value up to the self-resonant frequency. It breaks down most seriously when the wire diameter becomes comparable (of the order of 1/10th or more) with the coil diameter.

Nagaoka's inductance formula is usually written in the form

$$L_x = \frac{4\pi^2 R^2 n^2 K \, 10^{-9}}{l} \text{ henrys}$$

where R, l and n have the meanings previously defined, and K is a factor involving the ratio of length/diameter only.

Also, $R_x =$ (d.c. resist.). H . ϕ ohms.
where H has its high-frequency value (see Section 10) and ϕ is defined by Table VIII.

Hence,

$$R_x = \frac{2\pi R n}{\pi (d/2)^2}\,\rho \cdot \frac{1}{2\sqrt{2}}\,\pi d\sqrt{\frac{2f}{10^9\rho}}\,.\,\phi \text{ ohms.}$$

$$= \frac{\sqrt{2}\,Rn\,\rho}{\beta d}\,\phi \text{ ohms,}$$

where

$$\beta = \frac{1}{2\sqrt{2}\pi}\sqrt{\frac{10^9\rho}{f}}$$

Now,

$$Q = \frac{2\pi f\,L_x}{R_x}$$

$$= 2\pi f\,.\,\frac{4\pi^2 R^2 n^2}{l}\,K10^{-9}\,.\,\frac{\beta d}{Rn\rho\,\phi\,\sqrt{2}}$$

$$= \frac{\pi R}{\sqrt{2}\beta}\cdot\frac{nd}{l}\cdot\frac{K}{\phi}$$

$$= \frac{R}{\sqrt{2}\,\beta}\,\frac{\pi d}{s}\cdot\frac{K}{\phi}$$

$$= \frac{R}{\sqrt{2}\beta}\,\psi \text{ where } \psi \text{ is a function of}$$

d/s and l/D.

For copper, taking $\rho = 1.7 \times 10^{-6}$ ohm-cm we find that

$$Q = 0.15 R\,\psi\,\sqrt{f}$$

Table IX, which gives values of ψ for various values of coil length/diameter and spacing ratio, is derived from Table VIII and Nagaoka's table of K. The measured values of Q (uncorrected for leads) were checked against those predicted from this table. For coils falling within the body of the table, that is to say, to the left of the column length/diameter = 4, the difference was 5 per cent. or less, the values based on Table IX being usually higher than the experimental values. For coils with length/diameter = 5 or more, and d/s greater than 0.5, the measured values tended to be 10 per cent or more lower than the predicted values.

The discrepancy in the case of the long coils is due not to divergence of the measured resistances from the values corresponding to Table VIII but to inductance values different from those predicted by Nagaoka's formula. These coils all had small diameters, in order that a sufficiently large length/diameter ratio could be attained without excessive bulk of coil, and the wire diameter could no longer be regarded as small compared with the coil diameter. Now, Nagaoka's formula is a current-sheet formula and assumes that the thickness of this sheet

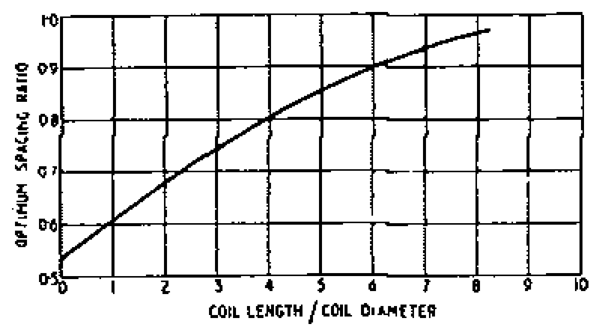

Fig. 12. *Variation of optimum spacing ratio with length/diameter.*

is negligible compared with the diameter. In using Nagaoka's formula, we have taken the mean diameter of our coil as the diameter of his equivalent current-sheet. However, the current in a coil, at high frequencies, tends to flow round the inner surface, so that the equivalent current-sheet should

TABLE IX.

d/s	Coil Length/Coil Diameter											
	0	0.2	0.4	0.6	0.8	1.0	2	4	6	8	10	∞
1.0	0.00	0.18	0.26	0.31	0.35	0.38	0.63	0.80	0.89	0.93	0.93	0.92
0.9	0.00	0.24	0.33	0.39	0.43	0.47	0.69	0.84	0.90	0.93	0.93	0.91
0.8	0.00	0.28	0.40	0.46	0.50	0.55	0.75	0.87	0.90	0.91	0.91	0.89
0.7	0.00	0.32	0.46	0.53	0.58	0.61	0.78	0.87	0.90	0.89	0.89	0.87
0.6	0.00	0.34	0.49	0.57	0.63	0.67	0.78	0.85	0.87	0.86	0.86	0.85
0.5	0.00	0.34	0.48	0.56	0.61	0.65	0.74	0.80	0.81	0.82	0.82	0.81
0.4	0.00	0.31	0.45	0.51	0.50	0.60	0.69	0.74	0.75	0.76	0.76	0.76
0.3	0.00	0.25	0.37	0.44	0.49	0.52	0.60	0.64	0.66	0.67	0.67	0.68
0.2	0.00	0.19	0.27	0.33	0.36	0.39	0.45	0.49	0.51	0.51	0.51	0.53
0.1	0.00	0.10	0.14	0.17	0.19	0.21	0.25	0.27	0.28	0.29	0.29	0.30

Values of ψ, from Table VIII and Nagaoka's inductance formula. High-frequency Q of a coil of copper wire or thick tubing is given by $Q = 0.15R\psi\sqrt{f}$.

have a diameter less than the mean diameter of the coil and greater than the inner diameter. That is to say, the measured inductance value should lie between the Nagaoka value obtained by using the mean diameter and that obtained by using the inner diameter. This is found to be the case.

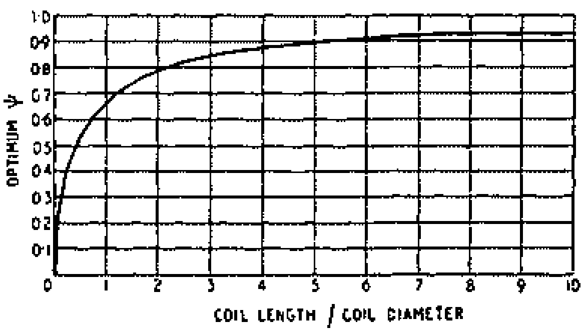

Fig. 13. *Variation of optimum ψ with length/diameter.*

Thus, for coil No. 54 (mean diameter 1.72 cm, wire diameter 0.12 cm), the two Nagaoka values corresponding to the inner and mean diameters respectively were 7.8 and 9.0 μH. The measured value was 8.15 μH., and, subtracting a (calculated) lead inductance of 0.13 μH., the coil inductance was 8.0 μH.

The entries in Table IX increase steadily with increasing length/diameter, except that when length/diameter approaches infinity, and $d/s > 0.4$, there is a small decline. This decline seems not to be readily explainable. The last three columns of the ϕ table, it will be remembered, are those resulting from Butterworth's theory when the transverse field term is neglected. The slight anomaly in the ψ table doubtless means that the effect of the transverse field is not quite negligible for length/diameter ratios of 8 and 10, when d/s is greater than 0.4.

The zero Q values for coils of zero length do not mean that the resistance is infinite, but that the inductance has disappeared.

For a given length/diameter, these entries show a rather flat optimum as the spacing ratio varies. In Fig. 12 the optimum spacing ratio is plotted against length/diameter. In Fig. 13 the value of ψ corresponding to the optimum spacing ratio is likewise plotted against length/diameter.

There is an interesting interpretation of ψ analogous to the interpretation of K in Nagaoka's formula. K may be defined as the ratio of the coil inductance to the inductance of an infinitely long cylindrical current sheet having a diameter equal to the mean diameter of the coil. K, in fact, is an end correction. Similarly, it can be shown from the results in reference 13 that ψ is the ratio of the coil Q to the Q at the same frequency of a certain idealized coil. This "coil" is an infinitely long cylinder, having its inner diameter equal to the mean diameter of the coil we are considering and a wall thickness large compared with the current penetration depth, the current being assumed to flow round the inner surface.

14. Low-frequency Resistance of Single-layer Coils.

When we derived a frequency correction to the measured h.f. resistance values, we assumed that the factor we have called k (in the version of Butterworth's formula given in Section 10) was independent of frequency if z was sufficiently high (of the order of 10 or more). This, it was pointed out, is not even approximately true in Butterworth's theory.

Another way of putting this is that, with close spacing of turns, in Butterworth's theory the h.f. resistance does not become proportional to the square root of the frequency until z is very high. When the turns are touching (physically, but not electrically), the h.f. resistance, in the theory, never becomes proportional to $\sqrt{f}$.

Butterworth gives values for his various quantities for z values up to 5, and for infinite z. Table I is based on these latter values. Interpolation for z values between 5 and ∞ is most conveniently done by plotting Butterworth's functions against the reciprocal of z. The values so obtained are, as one might expect, in closer agreement with the experimental results than those of Table I. Thus, when $z = 10$, for length/diameter = 1 we have the results of Table X.

TABLE X.

d/s	ϕ	% excess over experimental values
1.0	10.37	87%
0.9	5.57	36%
0.8	3.61	14%
0.7	2.61	6%

In the case of the single coil with $d/s = 0.95$, the theoretical value thus obtained was about 60 per cent. in excess of the measured value.

Thus, if Butterworth's low-frequency values can be relied upon, his predicted resistances are not so wildly in disagreement with experimental results as appears by comparison of Tables I and VIII, especially since in the top left-hand region of the Table, where the discrepancy will be largest, the coils, for physical reasons, had to be constructed with low z values, between 8 and 10.

We can easily show that these low-frequency Butterworth results are open to considerable suspicion. In fact, we shall see that, in consideration of the degree of approximation tolerated by Butterworth, any agreement with observation must, for closely spaced coils, be regarded as in the nature of an accident.

It was stated in Section 3 that Butterworth worked out each of his three types of loss by solving a set of an infinite number of linear equations, each containing an infinite number of unknowns. He used a method of successive approximations.

Now, when the frequency is low, that is to say, for $z = 5$ or less, the amount of arithmetic involved in proceeding beyond the first approximation becomes very large indeed. Consequently, for these values of z, Butterworth uses the first approximation only.

One can only guess at the error this introduces. For the case of touching wires, when z is infinite, there is an infinite error involved if we take only the first approximation for the transverse field losses. That is to say, the entries in the first row of Table I would be decreased from infinity to a series of not too large finite values.

Whether the converse is true, that is, whether if one proceeded to a sufficiently large number of approximations for the case of touching wires at low frequency an infinite result would be obtained, must be a matter of conjecture. On physical grounds, if a coplanar system of an infinite number of infinitely long touching wires offers infinite impedance to a transverse field of very high frequency, it seems not unreasonable to suppose that it will also offer an infinite impedance to a low-frequency transverse field.

Consequently, for different reasons, the applicability of the Butterworth low-frequency formula to specific coils is open to as much doubt as that of his high-frequency formula.

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Radiocommunication Convention

The Institution of Electrical Engineers is holding a convention, covering the wartime activities in the field of radiocommunications, from 25th to 28th March. The convention will be opened by the President of the Board of Trade, Sir Stafford Cripps, at 5.30 p.m. on Tuesday, 25th March, and he will introduce an address by Colonel Sir Stanley Angwin, on "Telecommunications in War."

On the following days there are to be morning, afternoon and evening sessions at which papers covering naval, military, short and long distance, and pulse communications will be read. Propagation, radio components and future trends will also be covered in the convention.

At a further meeting at 5.30 p.m. on 2nd April there will be a paper on C. W. Navigational Aids.

Physical Society's Exhibition

The 31st Exhibition of Scientific Instruments and Apparatus is being held by the Physical Society on 9th–12th April in the Physics and Chemistry Departments of Imperial College, South Kensington, London, S.W.7.

Admission is by ticket only and is restricted to members of the Society from 10 a.m. to 1 p.m., but it is open to non-members from 2 p.m. to 9 p.m.